[2] Gilbarg, D., and Serrin, J., *Free Boundaries and Jets in the Theory of Cavitation*, forthcoming publication.

[3] Koebe, P., "Allgemeine Theorie der Riemannschen Mannigfaltigkeiten," *Acta Math.*, **50**, 27–157 (1927). See also Courant, R., *Dirichlet's Principle*, Interscience (to be published); this uses the terminology *Riemann domain.*

---

*ON SOME GENERALIZATIONS OF THE CAUCHY-FRULLANI INTEGRAL**

By A. M. Ostrowski

University of Basle, Switzerland; U. S. National Bureau of Standards; and University of California at Los Angeles

Communicated by E. U. Condon, July 28, 1949

1. A beautiful result attributed often to Frullani is contained in the general formula

$$\int_0^\infty \frac{f(at) - f(bt)}{t}\,dt = (f(\infty) - f(0)) \log \frac{a}{b} \ (a, b > 0), \qquad (1)$$

where $f(0) = \lim\limits_{x \downarrow 0} f(x)$, $f(\infty) = \lim\limits_{x \to \infty} f(x)$ and $f(x)$ is assumed L-integrable over any interval $0 < A \leq x \leq B < \infty$.[1]

2. Suppose that the integral

$$\int_0^A \frac{f(t)}{t}\,dt \qquad (2)$$

exists for any $A > 0$. Then the above formula can be replaced by

$$\int_0^\infty \frac{f(at) - f(bt)}{t}\,dt = f(\infty) \log \frac{a}{b} \ (a, b > 0) \qquad (3)$$

if $f(\infty) = \lim\limits_{x \to \infty} f(x)$ exists. It can be expected that this formula remains valid if $f(\infty)$ is replaced by appropriate *mean values.*

Suppose, for instance, that $f(x)$ is *periodic with period* p and the integral (2) exists for any $A > 0$. Then $f(\infty)$ can be replaced by $\frac{1}{p}\int_u^{u+p} f(x)dx$ and we get the following very useful formula

$$\int_0^\infty \frac{f(at) - f(bt)}{t}\,dt = \frac{1}{p}\int_0^{u+p} f(x)dx \ \log \frac{a}{b}. \qquad (4)$$

3. Thus we obtain from

$$\int_0^{\pi} |\tan x|^{\alpha} dx = \frac{\pi}{\cos \dfrac{\alpha\pi}{2}} \quad (0 < \alpha < 1),$$

the formula

$$\int_0^{\infty} (|\tan ax|^{\alpha} - |\tan bx|^{\alpha}) \frac{dx}{x} = \frac{1}{\cos \dfrac{\alpha\pi}{2}} \log \frac{a}{b} \quad (0 < \alpha < 1;\ a, b > 0). \tag{5}$$

Similarly, it follows from

$$\int_0^{\pi} \log |\cos x| dx = \pi \log {}^1/_2$$

that

$$\int_0^{\infty} \log \left|\frac{\cos ax}{\cos bx}\right| \frac{dx}{x} = \log {}^1/_2 \log \frac{a}{b} \ (a, b > 0). \tag{6}$$

4. The condition of the periodicity of $f(t)$ can be replaced by an essentially more general one, that of the existence of the mean value

$$M(f) = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(t) dt \tag{7}$$

and we obtain the following general theorem:

If the integral (2) *exists for any $A > 0$ and the mean value* (7) *exists, we have for all positive a and b*

$$\int_0^{\infty} \frac{f(at) - f(bt)}{t} dt = M(f) \log \frac{a}{b}. \tag{8}$$

5. To prove (8) we use the fact that

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \qquad (x > 0)$$

is continuous for all $x > 0$ and tends to $M(f)$ as $x \to \infty$. We have on integrating by parts

$$\int_{bA}^{aA} \frac{f(t)}{t} dt = \int_{bA}^{aA} \frac{(tF(t))'}{t} dt = F(aA) - F(bA) + \int_{bA}^{aA} \frac{F(t)}{t} dt = F(aA) - F(bA) + F(\zeta) \log \frac{a}{b}$$

where $bA < \zeta < aA$. It follows

$$\lim_{A \to \infty} \int_{bA}^{aA} \frac{f(t)}{t} dt = M(f) \log \frac{a}{b}.$$

On the other hand we have

$$\int_0^{aA}\frac{f(t)}{t}dt - \int_0^{bA}\frac{f(t)}{t}dt = \int_0^{A}\frac{f(at)}{t}dt - \int_0^{A}\frac{f(bt)}{t}dt = \int_0^{A}\frac{f(at)-f(bt)}{t}dt$$

and our theorem is proved.

6. The assumption of the existence of (2) can be decomposed into two parts: (1) $f(t)$ is L-integrable over any interval $0 < \epsilon \leq t \leq A < \infty$; (2) $\frac{f(t)}{t}$ is L-integrable over any interval $0 \leq t \leq \epsilon$. The second part can be replaced by the assumption that $\lim\limits_{\epsilon\downarrow 0}\int_\epsilon^A \frac{f(t)}{t}dt$ exists or by the hypothesis that $f(0) = \lim\limits_{\epsilon\downarrow 0} f(t)$ exists and has the value 0. Both hypotheses are only special cases of a more general one which is obtained from the assumption about the behavior of $f(t)$ at $t = \infty$ by the transformation $t = \frac{1}{\tau}$. We come thus to the assumption

$$\epsilon\int_\epsilon^1 \frac{f(t)}{t^2}dt \to 0 \qquad (\epsilon \downarrow 0).$$

More generally if we assume that together with (7)

$$m(f) = \lim_{\epsilon\downarrow 0}\epsilon\int_\epsilon^1\frac{f(t)}{t^2}\,dt \tag{9}$$

exists, we have

$$\int_0^\infty \frac{f(at)-f(bt)}{t}dt = (M(f) - m(f))\log\frac{a}{b}. \tag{10}$$

7. The result obtained in (10) is in a certain sense the best result obtainable. It can be shown that if the integral at the left in (10) exists for some pairs of positive values a, b such that $\frac{b}{a}$ runs through a set of positive measure, then both limits $M(f)$, $m(f)$ exist and we have (10). However, the proof of this result is difficult and will be given in another publication.

Since finding the formula (10) I have discovered that the problem of convergence of the integral at the left in (10) has already been investigated and solved by K. S. K. Iyengar.[7] The necessary and sufficient conditions given by Iyengar consist in the existence of the four following limits:

$$\lim_{x\to\infty}\int_1^x\frac{f(t)}{t^2}dt, \qquad \lim_{x\to\infty} x\int_x^\infty \frac{f(t)}{t^2}\,dt \tag{11}$$

$$\lim_{x \downarrow 0} \int_x^1 f(t)dt, \qquad \lim_{x \downarrow 0} \frac{1}{x}\int_0^x f(t)dt. \tag{12}$$

It can be directly proved that both conditions (11) are equivalent to the existence of $M(f)$. Similarly, both conditions (12) are equivalent to the existence of $m(f)$. Iyengar's proof of his theorem and the simplified proof of it given by Agnew[2] are, however, still difficult since in both proofs certain theorems about non-uniform convergence are used which, although usually proved only in the case of convergent *sequences*, have to be used in the case of *continuous approximation*. Our proof of the necessity of the existence of $M(f)$ and $m(f)$ makes use of the theorem of Osgood, but exactly in the form proved by Osgood, that is for the case of *sequences* of functions.

8. Formula (1) can be considerably generalized by replacing *at* and *bt* by functions of t which to a certain extent are arbitrary. In this way we obtain a "three-function formula" giving the value of definite integrals containing three "arbitrary" functions.

This general formula cannot be directly reduced to (1) by substituting a new variable of integration. However, the interval of integration and the range of values of two of the functions can be reduced by such a transformation to the interval $(0, \infty)$. We obtain thus the canonical form of our formula

$$\int_0^\infty (\psi' g(\psi) - \varphi' g(\varphi))dx = M(xg(x)) \log\left(\frac{\psi}{\varphi}\right)_{x=\infty} - m(xg(x)) \log\left(\frac{\psi}{\varphi}\right)_{x=0}. \tag{13}$$

The functions $\varphi(x)$ and $\psi(x)$ are supposed in (13) to be positive and absolutely continuous for $0 < x < \infty$ and to tend to 0 with $x \downarrow 0$ and to ∞ with $x \to \infty$. We assume further that with $x \downarrow 0$ and $x \to \infty$ the limits of $\dfrac{\varphi(x)}{\psi(x)}$ exist and are positive. The function $g(x)$ is assumed to be integrable over any closed interval of the positive x-axis and such that $M(xg(x))$ and $m(xg(x))$ exist.

9. Three special cases of (13) have been previously given. In 1823 Cauchy[3] gave the formula

$$\int_0^1 \left(\frac{\psi' f(\psi)}{1-\psi} - \frac{\varphi' f(\varphi)}{1-\varphi}\right) dx = f(1) \log \frac{\varphi'(1)}{\psi'(1)}. \tag{14}$$

Here $f(x)$ is continuous in the closed interval $< 0, 1 >$; φ and ψ are differentiable in $< 0, 1 >$ with positive derivatives and it is assumed that $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$. In 1841 Cauchy[4] published another formula

$$\int_0^\infty \left(\psi' \frac{f(\psi)}{\psi} - \varphi' \frac{f(\varphi)}{\varphi}\right) dx = f(0) \log \frac{\varphi'(0)}{\psi'(0)}. \tag{15}$$

In this formula φ and ψ are positive in $(0, \infty)$ and have continuous derivatives which are positive at 0. Further we assume that $\varphi(0) = \psi(0) = 0$ and that as $x \to \infty$ $\varphi(x)$ and $\psi(x)$ tend to ∞. About $f(x)$ Cauchy assumes continuity for all $x \geq 0$. However, about the behavior of $f(x)$ for $x \to \infty$ Cauchy erroneously gives the condition $f(x) \to 0$. This condition is not sufficient for the validity of the above formula but Cauchy's argument remains valid if the existence of $\int_1^\infty \frac{f(x)}{x} dx$ is assumed.

Finally, 1891, M. Lerch[6] gave the formula

$$\int_a^b (g(x) - \varphi' g(\varphi(x)))dx = \\ ((x - a)g(x))_{x=a} \log \varphi'(a) - ((x - b)g(x))_{x=b} \log \varphi'(b) \tag{16}$$

that is obtained from (13) by obvious specializations and transformations.

Some of the results given in this note have been discovered independently by Professor Tricomi of the California Institute of Technology in his work on the Bateman manuscript project and are going to be published in the *Bulletin of the American Mathematical Society.* Professor Tricomi discovered our formula (4); further, a case of our formula (8), namely the case in which

$$\int_0^x f(x)dx - xM(f)$$

is $O(1)$—while in our theorem it is assumed that this difference is $o(x)$; as to the three-function formula, he finds the special case in which $g(x)$ has finite limits at 0 and ∞, respectively, and $xg(x)$ is periodic.

* The preparation of this paper was sponsored (in part) by the Office of Naval Research.

[1] This formula was first published by Cauchy in 1823,[3] and more completely 1827[3a] with a beautiful proof which is used as standard in all textbooks today. About 1829 Frullani[5] published the same formula and mentioned that he had communicated it to plana in 1821. However, Frullani's proof is completely illusory.

[2] Agnew, R. P., "Limits of Integrals," *Duke Math. J.*, **9**, 10–19 (1942).

[3] Cauchy, A., *J. École Polytech. (Paris)*, **12** (1823); *Oeuvres compl.* (2), **I**, 335–339.

[3a] Cauchy, A., *Exercises de Mathematiques*, 1827; *Oeuvres compl.* (2), **VII**, 157.

[4] Cauchy, A., *Exercises Analyse*, **2** (1841); *Oeuvres compl.* (2), **XII**, 416–417.

[5] Frullani, G., "Sopra Gli Integrali Definiti (Ricevuta adi 21 Novembre (1829)," *Memorie della Societa Italiana delle Socienze*, **20**, 448–467 (1828).

[6] Lerch, M., "Sur une extension de la formule de Frullani," *Verhandl. Prager Akad., math.-phys. Klasse* I_2, 123–131 (1891).

[7] Iyengar, K. S. K., "On Frullani Integrals," *J. Indian Math. Soc.* (2), **4**, 145–150 (1940); reprinted in *Proc. Cambridge Phil. Soc.*, **37**, 9–13 (1941).